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Noncommutativity and Transition Probability in Quantum Mechanics

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Using an operation that behaves as a noncommutative conjuction in orthomodular lattices, a way to define the transition probability for arbitrary quantum logics is given.

In quantum mechanics there is a lattice structure that appears naturally and it is that of a *complete orthomodular lattice.* This is just a complete lattice $(\mathbf{L}, \leq, \wedge, \vee)$ which satifies all of the following:

(I) There exists an orthocomplement $\perp: \mathbf{L} \to \mathbf{L}$ such that:
  (i) $a \leq b$ iff $b^{\perp} \leq a^{\perp}$.
  (ii) $(a^{\perp})^{\perp} = a$.
  (iii) $a^{\perp} \vee a = 1$.
  (iv) $a^{\perp} \wedge a = 0$.

(II) Given $a \leq b$ in $\mathbf{L}$, then $b = a \vee (b \wedge a^{\perp})$ or equivalently $a = (a \vee b^{\perp}) \wedge b$. This property is usually referred to as weak modularity.

In Román and Rumbos (1982) it is argued that if the implication $a \to b = (a \wedge b) \vee a^{\perp}$, known in the literature as the *Sasaki hook*, is chosen in order to turn $\mathbf{L}$ into an implicative lattice, then the best choice for the logical conjunction is the *ampersand* & defined by $a \,\&\, b = (a \vee b^{\perp}) \wedge b$. It is not hard to see that $a \,\&\, b \leq c$ iff $a \leq b \to c$, so that $\_ \,\&\, b$ is left adjoint to $b \to \_$.

It is worth mentioning that & is neither commutative nor associative; actually it is only so when $\& = \wedge$ and this happens if and only if $\mathbf{L}$ is a Boolean algebra. For a proof the reader is referred to Román and Rumbos (1991).

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Another condition relating Booleaness of **L** to properties of & is the following:

Proposition. Let **L** be an orthomodular lattice and $b \in \mathbf{L}$. The map $\_ \,\&\, b$ preserves orthogonality (i.e., $a^{\perp} \,\&\, b \leq (a \,\&\, b)^{\perp}\ \forall a \in \mathbf{L}$) iff **L** is Boolean.

Proof. $a^{\perp} \,\&\, b \leq (a \,\&\, b)^{\perp}$ iff $a^{\perp} \leq b \to (a \,\&\, b)^{\perp} = [b \wedge (a \,\&\, b)^{\perp}] \vee b^{\perp}$, but since $a \,\&\, b \leq b$ we have $b^{\perp} \leq (a \,\&\, b)^{\perp}$ and weak modularity yields $[b \wedge (a \,\&\, b)^{\perp}] \vee b^{\perp} = (a \,\&\, b)^{\perp}$; thus $a^{\perp} \,\&\, b \leq (a \,\&\, b)^{\perp}$ iff $a^{\perp} \leq (a \,\&\, b)^{\perp}$ iff $a \,\&\, b \leq a$ iff $a \,\&\, b = a \wedge b$ iff **L** is Boolean. ■

The ampersand behaves very much like a noncommutative analog of the usual meet in **L**. Two elements $a, b \in \mathbf{L}$ will be said to be *compatible* (denoted by $a \to b$) iff $a \,\&\, b = a \wedge b$.

In general, a *quantum logic* (Jauch, 1968) is a pair $(\mathbf{L}, \mathbf{S})$, where **L** is a complete orthomodular lattice and **S** is a full set of states. A *state* $s \in \mathbf{S}$ is just a map $s\colon \mathbf{L} \to [0, 1]$ satisfying:

(i) $s(0) = 0$, $s(1) = 1$.
(ii) $s(\vee a_i) = \sum s(a_i)$ given $a_i \leq a_j^{\perp}\ \forall i \neq j$.

Moreover, **S** is *full* whenever $s(a) \leq s(b)\ \forall s \in \mathbf{S} \Rightarrow a \leq b$.

If the Borel sets of $\mathbb{R}$ are denoted as usual by $\mathscr{B}(\mathbb{R})$, an **L**-*observable* (or just an *observable* when no confusion arises) is an **L**-valued measure based on $\mathscr{B}(\mathbb{R})$, that is, a map $\mathbf{A}\colon \mathscr{B}(\mathbb{R}) \to \mathbf{L}$ such that $\mathbf{A}(\emptyset) = 0$, $\mathbf{A}(\mathbb{R}) = 1$, and $\mathbf{A}(\bigcup \beta_i) = \sum \mathbf{A}(\beta_i)$ given $\beta_i \cap \beta_j = \emptyset$ for all $i \neq j$.

The orthomodular lattice associated with quantum mechanics is just the lattice of closed subspaces of a Hilbert space $\mathscr{H}$, or equivalently, the lattice of projection operators, which will be denoted by $\mathscr{P}(\mathscr{H})$. Note that when $\mathbf{L} = \mathscr{P}(\mathscr{H})$, an observable is a spectral measure and by the spectral theorem these correspond exactly to the self-adjoint operators on $\mathscr{H}$. In particular, the eigenvalues or possible measurements of an observable will always be real and they can form either a continuous set, if, for example, the observable is the position of a particle, or a discrete set, if we are dealing with an observable such as energy.

Given an observable $\mathbf{A}\colon \mathscr{B}(\mathbb{R}) \to \mathbf{L}$, a state $s\colon \mathbf{L} \to [0, 1]$, and a set $\beta \in \mathscr{B}(\mathbb{R})$, the number $s(\mathbf{A}(\beta)) \in [0, 1]$ is interpreted as "the probability that the observable **A** takes values in the set β when the system is in the state s."

If **A** and **B** are two observables, they are said to be *compatible* or to commute, iff $\mathbf{A}(\beta) \to \mathbf{B}(\gamma)$ for all β, $\gamma \in \mathscr{B}(\mathbb{R})$. When $\mathbf{L} = \mathscr{P}(\mathscr{H})$ and the observables are thought of as self-adjoint operators, then this is equivalent to $\mathbf{AB} = \mathbf{BA}$ and so their product (regular composition) is also an observable. This, of course, happens only when **A** and **B** can be simultaneously diagonalized, that is, when an orthonormal basis of common eigenvectors can be found for **A** and **B**.

Again let $\mathbf{L}=\mathcal{P}(\mathcal{H})$; if $\langle\cdot,\cdot\rangle$ is the scalar product on $\mathcal{H}$ and $\mathcal{U}$ denotes the set of all unit vectors, a full set of states is given by

$$\mathbf{S}=\{s_u: \mathcal{P}(\mathcal{H})\to[0,1] \mid u\in\mathcal{U} \text{ and } s_u(P)=\langle u, P(u)\rangle\ \forall P\in\mathcal{P}(\mathcal{H})\}$$

Note that if P is one-dimensional, say $P=P_v$, the projection onto the space generated by $v\in\mathcal{U}$, then $s_u(P_v)=|\langle u, v\rangle|^2$. This is usually referred to as the *transition probability* between the states s_u and s_v.

If **A** and **B** are two self-adjoint operators or observables which are not compatible, then there exist vectors $u, v\in\mathcal{U}$ such that u is an eigenvector of **A** but not of **B** and dually with v, thus $|\langle u, v\rangle|^2\in(0,1)$, that is, there is a nontrivial transition probability between the corresponding (eigen)states s_u and s_v. It is to be expected, then, that there should be a relationship between the noncommutativity of **A** and **B** and the transition probabilities of their noncommon (eigen)states.

The quantum logic approach to the foundations of quantum mechanics (e.g., Jauch, 1968) deals with an arbitrary quantum logic (**L**, **S**). One of the problems with this description is that it is static since it offers no reasonable (or unreasonable!) definition of transition probability between states; a description of this problem can be found in Gudder (1978). Maczynski (1981) attempts to solve this problem by defining a "measure of noncommutativity" for the elements of L. Román and Rumbos (1991), making use of the operation & discussed before, defined a simpler and conceptually clearer measure of noncommutativity; let us briefly describe it.

Whenever (**L**, **S**) is a quantum logic and $a, b\in\mathbf{L}$, the *commutativity gap* between a and b is defined as follows.

Definition 1. $\Delta(a,b)=\sup_{s\in S}|s(a \,\&\, b)-s(b \,\&\, a)|$.

It is immediate from the definition that Δ satisfies:

(i) $\Delta(a,b)=\Delta(b,a)$.
(ii) $\Delta(a,b)\in[0,1]$.
(iii) $\Delta(a,b)=0$ iff $a\to b$.

This definition can be easily extended to arbitrary observables as follows.

Definition 2. Given **A** and **B** two **L**-observables, then the commutativity gap between them is given by

$$\Delta(\mathbf{A},\mathbf{B})=\sup_{E,F\in B(R)}\Delta(\mathbf{A}(E),\mathbf{B}(F))$$

If $\mathbf{L}=\mathcal{P}(\mathcal{H})$ it was shown in Román and Rumbos (1991) that $\Delta(a,b)=\|a \,\&\, b-b \,\&\, a\|$, where $\|\cdot\|$ is the usual operator norm; also, if $u, v\in\mathcal{U}$ with $\langle u, v\rangle\neq 0$ and P_u, P_v are the corresponding one-dimensional projections, the following was proved: $\Delta(P_u, P_v) = (1 - |\langle u, v\rangle|^2)^{1/2}$. This expression will be referred to as $(*)$.

Given a quantum logic $(\mathbf{L}, \mathbf{S})$, it is usually required that $\mathbf{L}$ be atomic and that there exists a bijection between the atoms of $\mathbf{L}$ and the pure states of $\mathbf{S}$ ($s \in \mathbf{S}$ is *pure* iff s cannot be expressed as a convex combination of other elements of $\mathbf{S}$). Whenever this happens we shall say that the quantum logic is *desirable*. Expression $(*)$ above suggests the following definition:

Definition 3. Let $(\mathbf{L}, \mathbf{S})$ be a desirable quantum logic, let a, b be two atoms of $\mathbf{L}$, and let s_a, s_b be the corresponding pure states. The *transition probability* between s_a and s_b, denoted by $\operatorname{trp}(s_a, s_b)$, is defined by

$$\operatorname{trp}(s_a, s_b) = \begin{cases} 1 - \Delta(a, b)^2 & \text{if} \quad a \not\leq b^{\perp} \\ 0 & \text{if} \quad a \leq b^{\perp} \end{cases}$$

It is this definition that I suggest toward the solution of the open problem posed in Gudder (1978) mentioned above.

Let us now go back to the case $\mathbf{L} = \mathscr{P}(\mathscr{H})$. Given $\mathbf{A}, \mathbf{B}$ self-adjoint operators, their commutator $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ is not self-adjoint, but the operator $i[\mathbf{A}, \mathbf{B}]$ is. It is natural to ask whether, given $u, v \in \mathscr{U}$ and p_u, p_v the corresponding projections, there is any connection between $\|i[p_u, p_v]\|$ and $\Delta(p_u, p_v)$. The answer is given by the next result.

Theorem. Given p_u, p_v as above with $\langle u, v\rangle \neq 0$, we have that

$$\Delta(p_u, p_v) = \frac{\|i[p_u, p_v]\|}{|\langle u, v\rangle|}$$

Proof. Expressing p_u and p_v in the orthonormal basis

$$e_1 = u, \qquad e_2 = \frac{v - \langle v, u\rangle u}{(1 - |\langle u, v\rangle|^2)^{1/2}}$$

as in Román and Rumbos (1991) we have that

$$p_u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad p_v = \begin{pmatrix} |\langle u, v\rangle|^2 & \langle v, u\rangle(1 - |\langle u, v\rangle|^2)^{1/2} \\ \langle u, v\rangle(1 - |\langle u, v\rangle|^2)^{1/2} & 1 - |\langle u, v\rangle|^2 \end{pmatrix}$$

so that $\|i[p_u, p_v]\|$ can be calculated as the largest eigenvalue of $i(p_u p_v - p_v p_u)$. This is easily seen to be $|\langle u, v\rangle|(1 - |\langle u, v\rangle|^2)^{1/2}$, where we recognize $\Delta(p_u, p_v)$ as the right-hand side factor; we thus have that $\Delta(p_u, p_v) = \|i[p_u, p_v]\| / |\langle u, v\rangle|$, which is the desired result. ■

The advantage of defining the transition probability in terms of the commutativity gap Δ is that this can be done for more general quantum logics $(\mathbf{L}, \mathbf{S})$, where the notion of commutator does not even exist.

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